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The relation between invariant integrals of the linear isotropic theory of elasticity and integrals defined by the duality principle☆

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ABSTRACT

Invariant integrals of the linear isotropic theory of elasticity, determined by a certain specified elastic field, are considered, and also invariant integrals generated by the interaction of the specified field with an arbitrary secondary field. For all types of invariant integral, generated by the interaction of the specified elastic field and an arbitrary secondary elastic field, transformations of the secondary fields are found for which the invariant integrals considered turn out to be equal to the *RG*-integrals, determined by the duality principle, of the specified elastic field and the transformed secondary elastic field. The invariant *J*-, *L*- and *M*-integrals themselves are also expressed in terms of the *RG*-integrals of the specified elastic field and its corresponding transformation.

Invariant *J*-, *L*- and *M*-integrals[1,2] are widely used in fracture mechanics to determine the elastic energy release rate during crack propagation and for calculating stress intensity factors. Invariant integrals, depending on two stress states of the elastic body and characterizing the interaction of these states, were introduced.[3] Such integrals are a generalization of integrals introduced earlier[1,2] and are used to separate the stress intensity factors, combinations of which are employed to express the elastic energy release rate.[4–8] Similar problems are also solved using integrals determined by the duality principle.[9,10]

The *RG*-integrals, determined by the duality principle, and various types of invariant integral began to be used comparatively recently when solving inverse problems of defect identification in an elastic body.[11–14] Since various integrals are used when solving problems of the same type and lead to similar results, the question arises as to the relation between these integrals. For the plane problem of the theory of elasticity and a defect comprising a rectilinear crack, the relation between *RG*-integrals and invariant *J*- and *M*-integrals was established.[15,16] For the three-dimensional problem of the theory of elasticity and arbitrary defects, the relation between *RG*-integrals and *M*-integrals characterizing the interaction of a specified elastic field with regular fields of special form was established and demonstrated.[12,14] A detailed review of results concerning invariant integrals and their application is available.[17]

The aim of the present paper is to establish the relation in the general case between the *J*-, *L*- and *M*-integrals, and also the integrals of interaction generated by these invariant integrals and the corresponding *RG*-integrals.

1. Formulation of the problem

Consider a linearly elastic isotropic body with a shear modulus μ and Poisson's ratio ν, occupying the region $\Omega = V\backslash\bar{G}$, where $V \subset R^3$ is a simply connected region, $G \subset V$ is a finite set, which below will be referred to as the defect, and $\bar{G} \subset V$, $\bar{G}$ is the closure of the set G. The defect G may be a cavity, a crack, an inclusion, etc. We will introduce the Cartesian system of coordinates $Ox_1x_2x_3$. The stress–strain state in the body Ω will be denoted by the superscript f: σ_{ij}^f is the stress tensor, e_{ij}^f is the strain tensor and $\mathbf{u}^f = (u_1^f,\ u_2^f,\ u_3^f)$ is the displacement

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vector. In the light of our assumptions concerning the material of body Ω, the following equations hold:

$$
\begin{aligned}
&e^f_{ij}(x) = (u^f_{i,j}(x) + u^f_{j,i}(x))/2, \quad i = 1,2,3; \quad j = 1,2,3\\
&\sigma^f_{ij}(x) = 2\mu\left[\frac{\nu}{1-2\nu}\theta^f(x)\delta_{ij} + e^f_{ij}(x)\right], \quad \theta^f(x) = \sum_{k=1}^{3} e^f_{kk}(x)\\
&\sigma^f_{ij,j}(x) = 0, \quad x = (x_1, x_2, x_3) \in \Omega
\end{aligned}
\tag{1.1}
$$

Here and below, summation over repeated indices is implied, and δ_{ij} is the Kronecker delta. Besides the specified stress state, we will also consider secondary stress states, which will be denoted by the superscript r. The corresponding elastic field can satisfy Eqs (1.1) both in the entire region V (i.e., can be regular) and only in the region Ω occupied by the body.

Consider the integral

$$
RG^f(r) = \int_S (t^f_i u^{r'}_i - t^r_i u^f_i)dS; \quad t^f_i = \sigma^f_{ij}(x)n_j(x), \quad t^r_i = \sigma^r_{ij}(x)n_j(x)
\tag{1.2}
$$

where S is a closed surface, and $\mathbf{n} = (n_1(x), n_2(x), n_3(x))$ is the unit vector of the outward normal to surface S. It follows from the duality principle that $RG^f(r) = 0$ when the surface S does not contain the set G and $S \cap \bar{G} = \varnothing$. In the opposite case, the quantity $RG^f(r)$ may be non-zero, and its value gives some information about the defect G. Note that, for all surfaces S containing the region G, such that $S \cap \bar{G} = \varnothing$, the values of the integrals $RG^f(r)$ will be identical. These properties of the RG-integrals were used earlier[11] to identify a plane crack.

From the invariance of the equations of elasticity theory in relation to translations, rotations and similarity transformations, the following invariant integrals were obtained[1]

$$
\begin{aligned}
&J_i = \int_S (Wn_i - t_j u_{j,i})dS\\
&L_i = \int_S \varepsilon_{ijk}(Wx_k n_j + t_j u_k - t_p u_{p,j} x_k)dS; \quad i = 1,2,3\\
&M = \int_S \left(Wx_i n_i - t_j u_{j,i} x_i - \frac{1}{2} t_i u_i\right)dS
\end{aligned}
\tag{1.3}
$$

which possess the same properties as the integrals $RG^f(r)$. Here, σ_{ij}, e_{ij} and $\mathbf{u} = (u_1, u_2, u_3)$ are the stress tensor, the strain tensor and the displacement vector, respectively, for a certain stress–strain state in the elastic body, $W = \sigma_{kl}e_{kl}/2$, $t_i = \sigma_{ij}n_j$ and ε_{ijk} is the Levi-Civita symbol.

All integrals defined by Eq. (1.3) are zero if the closed surface S contains no defects. If any defect lies inside the surface S, the invariant integrals may be non-zero, and their values provide information on the defect. Thus, all the invariant integrals (1.3) can be used to identify the defect in a similar way to the use of the duality principle (1.2).

A superscript f will denote the invariant integrals for the elastic field $\mathbf{u}^f$: J^f_i, L^f_i and M^f. A superscript r will denote the invariant integrals for the elastic field $\mathbf{u}^r$: J^r_i, L^r_i and M^r. Note that, when the field $\mathbf{u}^r$ is regular, all the integrals J^r_i, L^r_i and M^r are zero; otherwise, they are generally non-zero. We will consider the invariant integrals for the sum of fields with superscripts f and r, which will be denoted by a superscript $f+r$. For the overall field we have

$$
J^{f+r}_i = J^f_i + J^r_i + J^f_{i\text{int}}(r), \quad L^{f+r}_i = L^f_i + L^r_i + L^f_{i\text{int}}(r), \quad M^{f+r} = M^f + M^r + M^f_{\text{int}}(r)
\tag{1.4}
$$

Integrals corresponding to the interaction between fields with superscripts f and r have the form

$$
\begin{aligned}
&J^f_{i\text{int}}(r) = \int_S (\sigma^f_{kl}e^r_{kl}n_i - t^f_j u^r_{j,i} - t^r_j u^f_{j,i})dS\\
&L^f_{i\text{int}}(r) = \int_S \varepsilon_{ijk}(\sigma^f_{mn}e^r_{mn}x_k n_j + t^f_j u^r_k + t^r_j u^f_k - t^f_p u^r_{p,j}x_k - t^r_p u^f_{p,j}x_k)dS\\
&M^f_{\text{int}}(r) = \int_S \left(\sigma^f_{kl}e^r_{kl}x_i n_i - t^f_j u^r_{j,i}x_i - t^r_j u^f_{j,i}x_i - \frac{1}{2}t^f_i u^r_i - \frac{1}{2}t^r_i u^f_i\right)dS
\end{aligned}
\tag{1.5}
$$

Integrals (1.5) are also invariant; they were introduced earlier[3] and used[12,13] to solve the problem of identifying a spherical cavity and an elastic spherical inclusion.

Below, a slightly different notation of integrals (1.5) will be used, taking into account the equation $\sigma^f_{kl}e^r_{kl} = \sigma^r_{kl}e^f_{kl}$. We will show that, for each of the integrals represented by Eq. (1.5), a transformation of the field with superscript r will be found such that the corresponding integral will be equal to the RG-integral of the field with superscript f and the transformed field.

2. The relation between the *J*- AND *RG*-integrals

Let the vector function $\mathbf{u}^r = (u_1^r(x),\ u_2^r(x),\ u_3^r(x))$ satisfy Eq. (1.1) either in the region V or the region Ω. Consider the vector functions

$$\mathbf{u}^{D_k r} = (u_{1,k}^r(x), u_{2,k}^r(x), u_{3,k}^r(x)), \quad k = 1, 2, 3 \tag{2.1}$$

Since the system of Eq. (1.1) is invariant to translation, the vector functions $u^{D_k r}(x)$ satisfy Eq. (1.1) in the same region as the vector function $u^r(x)$

Theorem 1. *The following equation holds*

$$J_{k\,\mathrm{int}}^f(r) = -RG^f(D_k r) \tag{2.2}$$

Proof. To be spcific, we will consider the case $k=1$. First, suppose we have a cube K with centre at the point $x^0 = (x_1^0,\ x_2^0,\ x_2^0)$ and with sides $2L\,(K = \{x : |x_i - x_i^0| \le L, i = 1, 2, 3\})$ such that $\bar{G} \subset K \subset V, \bar{G} \cap \partial K = \varnothing$ and ∂K is the surface of the cube K. Since the integrals considered do not depend on the closed surface enveloping region G, we will select the cube surface ∂K as the surface S. For the cube faces we will introduce the notation

$$S_i = \{x : x \in \partial K, x_i = x_i^0 + L\}, \quad S_{i+3} = \{x : x \in \partial K, x_i = x_i^0 - L\}, \quad i = 1, 2, 3$$

and for the integrals over these faces

$$RJ_p = \int_{S_p} (t_i^f u_{i,1}^r - \sigma_{ij,1}^r n_j u_i^f) dS, \quad JI_p = \int_{S_p} (\sigma_{kl}^r e_{kl}^f n_1 - t_j^f u_{j,1}^r - t_j^r u_{j,1}^f) dS \tag{2.3}$$

From relations (1.2), (1.5) and (2.3) we have

$$RG^f(D_1 r) = \sum_{p=1}^{6} RJ_p, \quad J_{1\,\mathrm{int}}^f(r) = \sum_{p=1}^{6} JI_p \tag{2.4}$$

We will consider successively the integrals over all cube faces. Taking into account that, on opposite faces, the calculation of the integrals is quite similar, it is sufficient to consider the integrals over the faces S_1, S_2 and S_3.

From Eqs (2.3) we have

$$RJ_1 = \int_{S_1} (\sigma_{i1}^f u_{i,1}^r - \sigma_{i1,1}^r u_i^f) dS, \quad JI_1 = \int_{S_1} (\sigma_{kl}^r e_{kl}^f - \sigma_{j1}^f u_{j,1}^r - \sigma_{j1}^r u_{j,1}^f) dS \tag{2.5}$$

We will transform the expression for JI_1. From the second equation of (2.5) we obtain

$$JI_1 = \int_{S_1} (\sigma_{k2}^r u_{k,2}^f + \sigma_{k3}^r u_{k,3}^f - \sigma_{j1}^f u_{j,1}^r) dS$$

Integrating by parts, and, for the unit vector of the outward normal to the contour ∂S_p bounding the face S_p, introducing the notation

$$\nu^{(p)} = (\nu_1^{(p)}(x'), \nu_2^{(p)}(x'), \nu_3^{(p)}(x')), \quad x' \in \partial S_p$$

we obtain

$$JI_1 = \int_{S_1} (-\sigma_{k2,2}^r u_k^f - \sigma_{k3,3}^r u_k^f - \sigma_{j1}^f u_{j,1}^r) dS + \int_{\partial S_1} (\sigma_{k2}^r \nu_2^{(1)} + \sigma_{k3}^z \nu_3^{(1)}) u_k^f dl \tag{2.6}$$

From Eq. (2.6) and the equations of equilibrium using the first equation in system (2.5) we have

$$JI_1 = -RJ_1 + \int_{\partial S_1} (\sigma_{k2}^r \nu_2^{(1)} + \sigma_{k3}^z \nu_3^{(1)}) u_k^f dl \tag{2.7}$$

For the integrals over face S_2, according to expressions (2.3), after integration by parts in the second equation of (2.3) and introducing the notation

$$A_{pq} = \int_{\partial S_p} \sigma_{jq}^r \nu_1^{(p)} u_j^f dl$$

we will obtain

$$JI_2 = -RJ_2 - A_{22} \tag{2.8}$$

Similarly, for integrals over face S_3 we have

$$JI_3 = -RJ_3 - A_{33} \tag{2.9}$$

In a similar way, integrals evaluated over the faces S_4, S_5 and S_6 lead to the relations

$$JI_4 = -RJ_4 - \int_{\partial S_4} (\sigma^r_{k2}\nu^{(4)}_2 + \sigma^r_{k3}\nu^{(4)}_3)u^f_k dl, \quad JI_5 = -RJ_5 + A_{52}, \quad JI_6 = -RJ_6 + A_{63} \tag{2.10}$$

Adding Eqs. (2.7)–(2.10), we obtain formula (2.2) for $k = 1$, since the integrals over the edges of the cube cancel out.

Eq. (2.2) for $k = 2$ and $k = 3$ is proved in a similar way.

Thus, Eq. (2.2) is proved when, in region V, it is possible to inscribe a cube containing the defect G. In the general case, the following method is used to prove the theorem. In accordance with the accepted terminology,[18] we will refer to the set

$$K^{(n)} = \{x : m_i 10^{-n} \le x_i \le (m_i + 1)10^{-n}, \ i = 1, 2, 3\}$$

as a standard cube of rank n. Here, m_i are integers. Standard cubes of rank n can intersect only along a face, an edge or a vertex. The union of all cubes of rank n covers the space. We will put

$$Q^{(n)}_{\text{ext}}(G) = \bigcup_p K^{(n)}_p$$

where $\{K^{(n)}_p\}$ is the union of all standard cubes of rank n, such that $K^{(n)}_p \cap \bar{G} \neq \varnothing$. It is obvious that $\bar{G} \subset Q^{(n)}_{\text{ext}}(G)$ and $Q^{(n+1)}_{\text{ext}}(G) \subset Q^{(n)}_{\text{ext}}(G)$. A number $n = N$ is found such that $Q^{(N)}_{\text{ext}}(G) \subset V$. Consequently, $\bar{G} \subset Q^{(N)}_{\text{ext}}(G) \subset V$. We will choose the boundary $\partial Q^{(N)}_{\text{ext}}(G)$ of the set $Q^{(N)}_{\text{ext}}(G)$ as the surface S. This boundary consists of the faces of standard cubes of rank N, each of which is parallel to one of the coordinate planes. Intergrating over the faces, as was done above, we obtain one of the relations (2.7)–(2.10). Then, adding these relations together, we obtain formula (2.2), since the integrals over the edges cancel out.

Remark 1. The vector functions $\mathbf{u}^{D_k r}(x)$ satisfy Eqs (1.1), but, since the functions $u^r_i(x)$ have the dimension of displacements, the functions $\mathbf{u}^{D_k r}_i(x)$ are dimensionless. On the strength of this, formula (2.2) can be understood in the following way

$$J^f_{k\,\text{int}}(r) = -RG^f(lD_k r)/l$$

where l is the linear dimension and $\mathbf{u}^{lD_k r}(x) = l\mathbf{u}^{D_k r}(x)$.

Remark 2. From formulae (1.3) and (1.5) it follows that, if the initial field with superscript f is adopted as the field with superscript r, the following equation will be satisfied

$$J^f_{k\,\text{int}}(f) = 2J^f_k$$

From this, and from formula (2.2), we have

$$J^f_k = -RG^f(D_k f)/2 \tag{2.11}$$

3. The relation between the *M*- and *RG*-integrals

Consider the vector function

$$\mathbf{u}^{Hr}(x) = (u^{Hr}_1, u^{Hr}_2, u^{Hr}_3); \quad u^{Hr}_i = x_k u^r_{i,k}(x), \quad i = 1, 2, 3 \tag{3.1}$$

Since the elasticity theory equations (1.1) are invariant under a similarity transformation, the fact that the vector function $\mathbf{u}^r(x)$ satisfies system of equations (1.1) indicates[19] that the vector function $\mathbf{u}^{Hr}(x)$ satisfies system (1.1) in the same region. Consider the vector function

$$\mathbf{u}^{Pr}(x) = (u^{Pr}_1, u^{Pr}_2, u^{Pr}_3); \quad u^{Pr}_i = x_k u^r_{i,k}(x) + u^r_i(x)/2, \quad i = 1, 2, 3 \tag{3.2}$$

This also satisfies Eqs (1.1) in the same region as the vector function $\mathbf{u}^r(x)$.

Theorem 2. *The following equation holds*

$$M^f_{\text{int}}(r) = -RG^f(Pr) \tag{3.3}$$

Proof. The stresses corresponding to the displacements $\mathbf{u}^{Pr}(x)$ have the form

$$\sigma^{Pr}_{ij}(x) = x_k \sigma^r_{ij,k}(x) + 3\sigma^r_{ij}(x)/2 \tag{3.4}$$

From relations (1.2), (3.2) and (3.4) we have

$$RG^f(Pr) = \int_S \left[\sigma^f_{ij} n_j \left(x_k u^r_{i,k} + \frac{1}{2}u^r_i\right) - \left(x_k \sigma^r_{ij,k} + \frac{3}{2}\sigma^r_{ij}\right) n_j u^f_i\right] dS \tag{3.5}$$

Like the proof of Theorem 1, suppose initially that a cube K is found such that $\bar{G} \subset K \subset V$. We will introduce the the following notation for the integrals over the cube faces

$$RM_p = \int_{S_p} \left[\sigma^f_{ij} n_j \left(x_k u^r_{i,k} + \frac{1}{2} u^r_i\right) - \left(x_k \sigma^r_{ij,k} + \frac{3}{2}\sigma^r_{ij}\right) n_j u^f_i\right] dS$$

$$MI_p = \int_{S_p} \left(\sigma^r_{kl} e^f_{kl} x_i n_i - t^f_j u^r_{j,i} x_i - t^r_j u^f_{j,i} x_i - \frac{1}{2} t^f_i u^r_i - \frac{1}{2} t^r_i u^f_i\right) dS; \quad p = 1, 2, ..., 6 \tag{3.6}$$

From relations (1.5), (3.5) and (3.6) we obtain

$$RG^f(Pr) = \sum_{p=1}^{6} RM_p, \quad M^f_{\text{int}}(r) = \sum_{p=1}^{6} MI_p \tag{3.7}$$

Consider the quantities RM_1 and MI_1. We will rewrite the expression for MI_1 in the slightly different form

$$MI_1 = \int_{S_1} \left(\sigma^r_{k\alpha} u^f_{k,\alpha} x_1 - \sigma^f_{j1} u^r_{j,i} x_i - \sigma^r_{j1} u^f_{j,\alpha} x_\alpha - \frac{1}{2}\sigma^f_{i1} u^r_i - \frac{1}{2}\sigma^r_{i1} u^f_i\right) dS, \quad \alpha = 2, 3$$

Integrating by parts, we obtain

$$MI_1 = \int_{S_1} \left(-\sigma^r_{k\alpha,\alpha} u^f_k x_1 - \sigma^f_{j1} u^r_{j,i} x_i + \sigma^r_{j1,\alpha} u^f_j x_\alpha + \frac{3}{2}\sigma^r_{i1} u^f_i - \frac{1}{2}\sigma^f_{i1} u^r_i\right) dS +$$
$$+ \int_{\partial S_1} (\sigma^r_{k\alpha} u^f_k x_1 - \sigma^r_{j1} u^f_j x_\alpha) \nu^{(1)}_\alpha dl \tag{3.8}$$

From the equations of equilibrium it follows that $\sigma^r_{k\alpha,\alpha} = -\sigma^r_{k1,1}$. After using this equation in formula (3.8), and comparing the expression with the first equation in system (3.6) with $p = 1$, we obtain the equation

$$MI_1 = -RM_1 + \int_{\partial S_1} (\sigma^r_{k\alpha} u^f_k x_1 - \sigma^r_{j1} u^f_j x_\alpha) \nu^{(1)}_\alpha dl \tag{3.9}$$

Similar equations can be written for the other faces of the cube

$$MI_p = -RM_p + \int_{\partial S_p} (\sigma^r_{km} u^f_k x_p - \sigma^r_{jp} u^f_j x_m) \nu^{(p)}_m (1 - \delta_{pm}) dl, \quad m = 1, 2, 3, \quad p = 2, 3$$

$$MI_q = -RM_q - \int_{\partial S_q} (\sigma^r_{km} u^f_k x_q - \sigma^r_{jq} u^f_j x_m) \nu^{(q)}_m (1 - \delta_{qm}) dl, \quad q = 4, 5, 6 \tag{3.10}$$

In these formulae, summation over the indices p and q is not carried out.

Adding Eqs (3.9) and (3.10), we ensure that the integrals over the edges of the cube cancel out. On account of this, and taking Eqs (3.7) into account, we obtain Eq. (3.3).

In the general case, the proof of Theorem 2 is based on the same reasoning as the proof of Theorem 1. For this, the surface of the set $Q^{(N)}_{\text{ext}}(G)$ must be adopted as surface S.

Remark 3. From relations (1.3) and (1.5) we obtain the equation $M^f_{\text{int}}(f) = 2M^f$. From this equation and Eq. (3.3) we have

$$M^f = -RG^f(Pf)/2 = -RG^f(Hf)/2 \tag{3.11}$$

Remark 4. When $u^r_i(x)$ are homogeneous functions of the order of $m(u^r_i(kx) = k^m u^r_i(x))$, formula (3.3) is converted into the formula obtained earlier.[12,14]

Remark 5. Formulae similar to formulae (2.11) and (3.11) for the plane problem of the theory of elasticity and a defect in the form of a rectilinear crack were obtained earlier.[15,16]

4. Relations between the *L*- and *RG*-integrals

Consider the vector functions

$$\mathbf{u}^{Z_k r}(x) = (u_1^{Z_k r}, u_2^{Z_k r}, u_3^{Z_k r}), \quad u_m^{Z_k r} = \varepsilon_{kst} x_s u^r_{m,t}(x) + \varepsilon_{kms} u^r_s(x), \quad k = 1, 2, 3 \tag{4.1}$$

The equations of elasticity theory (1.1) are invariant under rotation. On the strength of this, the vector functions $\mathbf{u}^{Z_k r}(x)$ satisfy Eqs (1.1) in the same region as the vector functions $\mathbf{u}^r(x)$.[19] The stresses corresponding to the displacements $\mathbf{u}^{Z_k r}(x)$ have the form

$$\sigma_{mq}^{Z_k r} = \varepsilon_{kst}x_s\sigma_{mq,t}^r + \varepsilon_{kmt}\sigma_{qt}^r + \varepsilon_{kqt}\sigma_{mt}^r \tag{4.2}$$

Theorem 3. *The following equation holds*

$$L_{k\,\mathrm{int}}^f(r) = RG^f(Z_k r) \tag{4.3}$$

Proof. To be specific, we will consider the case when $k = 1$. Initially, as above, suppose a cube K exists such that $\bar{G} \subset K \subset V$. The surface of the cube will be chosen as the surface S. For the integrals over the cube faces we will introduce the notation

$$RL_p = \int_{S_p}(t_i^f u_i^{Z_1 r} - t_i^{Z_1 r}u_i^f)dS$$

$$LI_p = \int_{S_p}\varepsilon_{1jk}(\sigma_{mn}^r e_{mn}^f x_k n_j + t_j^f u_k^r + t_j^r u_k^f - t_q^f u_{q,j}^r x_k - t_q^r u_{q,j}^f x_k)dS \tag{4.4}$$

From relations (1.2), (1.5) and (4.4) we have

$$RG^f(Z_1 r) = \sum_{p=1}^{6} RL_p, \quad L_{1\,\mathrm{int}}^f(r) = \sum_{p=1}^{6} LI_p \tag{4.5}$$

We will consider successively the integrals (4.4) over all the cube faces. Since the integrals over opposite faces are evaluated in a similar way, it is sufficient to evaluate the integrals over the faces S_1, S_2 and S_3.

For the integrals over face S_1, from Eqs (4.1), (4.2) and (4.4) we obtain

$$RL_1 = \int_{S_1}[\sigma_{i1}^f(\varepsilon_{1st}x_s u_{i,t}^r + \varepsilon_{1is}u_s^r) - (\varepsilon_{1st}x_s\sigma_{i1,t}^r + \varepsilon_{1it}\sigma_{t1}^r)u_i^f]dS$$

$$LI_1 = \int_{S_1}\varepsilon_{1jk}(\sigma_{j1}^f u_k^r + \sigma_{j1}^r u_k^f - \sigma_{q1}^f u_{q,j}^r x_k - \sigma_{q1}^r u_{q,j}^f x_k)dS \tag{4.6}$$

Note that, in the second equation of (4.6), $j = 2, 3$, $k = 2, 3$ and $j \neq k$. Taking this into account and integrating by parts, we have

$$LI_1 = \int_{S_1}\varepsilon_{1jk}(\sigma_{j1}^f u_k^r + \sigma_{j1}^r u_k^f - \sigma_{q1}^f u_{q,j}^r x_k + \sigma_{q1,j}^r u_q^f x_k)dS - \varepsilon_{1jk}\int_{\partial S_1}\sigma_{q1}^r u_q^f x_k \nu_j^{(1)}dl \tag{4.7}$$

From Eqs. (4.6) and (4.7) we obtain

$$LI_1 = RL_1 - \int_{\partial S_1}\sigma_{q1}^z u_q^f(x_3\nu_2^{(1)} - x_2\nu_3^{(1)})dl \tag{4.8}$$

For the integrals over face S_2 we have

$$RL_2 = \int_{S_2}[\sigma_{i2}^f(\varepsilon_{1st}x_s u_{i,t}^r + \varepsilon_{1is}u_s^r) - (x_2\sigma_{i2,3}^r - x_3\sigma_{i2,2}^r + \varepsilon_{1it}\sigma_{t2}^r + \sigma_{i3}^r)u_i^f]dS$$

$$LI_2 = \int_{S_2}[\sigma_{mn}^r e_{mn}^f x_3 + \varepsilon_{1jk}(\sigma_{j2}^f u_k^r + \sigma_{j2}^r u_k^f - \sigma_{q2}^f u_{q,j}^r x_k - \sigma_{q2}^r u_{q,j}^f x_k)]dS \tag{4.9}$$

We will transform the second formula in system (4.9)

$$LI_2 = \int_{S_2}[\varepsilon_{1jk}(\sigma_{j2}^f u_k^r + \sigma_{j2}^r u_k^f - \sigma_{q2}^f u_{q,j}^r x_k) + \sigma_{m1}^r u_{m,1}^f x_3 + \sigma_{m3}^r u_{m,3}^f x_3 + \sigma_{q2}^r u_{q,3}^f x_2]dS \tag{4.10}$$

and introduce the notation

$$B_{\alpha\beta\gamma} = \int_{\partial S_\alpha}[(\sigma_{m1}^r\nu_1^{(\alpha)} + \sigma_{m\beta}^r\nu_\beta^{(\alpha)})u_m^f x_\beta + \sigma_{q\gamma}^r u_q^f x_\gamma \nu_\beta^{(\alpha)}]dl$$

Here, summation over α, β and γ is not carried out.

Integrating by parts on the right-hand side of Eq. (4.10), using the equations of equilibrium and the first equation of (4.9), we obtain

$$LI_2 = RL_2 + B_{232} \tag{4.11}$$

Similarly, for the integrals over face S_3 we have

$$LI_3 = RL_3 - B_{323} \tag{4.12}$$

The integrals over the remaining faces of the cube are obtained in a similar way:

$$LI_4 = RL_4 + \int\limits_{\partial S_4} \sigma^r_{q1} u^f_q (x_3 \nu_2^{(4)} - x_2 \nu_3^{(4)}) dl, \quad LI_5 = RL_5 - B_{532}, \quad LI_6 = RL_6 + B_{623} \tag{4.13}$$

Adding the integrals over all the cube faces, it can be shown that the integrals over the edges of the cube cancel out, and we arrive at formula (4.3). To prove formula (4.3), in the general case it is necessary, as above, to consider the surface of the set $Q^{(N)}_{\text{ext}}(G)$.

Remark 6. From relations (1.3) and (1.5) we have

$$L^f_{k\,\text{int}}(f) = 2L^f_k \tag{4.14}$$

From relations (4.3) and (4.14) we have

$$L^f_k = RG^f(Z_k f)/2 \tag{4.15}$$

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